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Uniqueness of Gibbs State for Nonideal Gas in $\mathbb{R}^d$: The Case of Pair Potentials

E. Pechersky[1] **and Yu. Zhukov**[1]

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We consider a classical gas of particles in $\mathbb{R}^d$ interacting via a pair potential. We prove that in a given region of the (β, μ) plane, where β is the inverse temperature, and μ is the chemical potential, either the Gibbs state is unique or it does not exist. Our method uses a version of the well-known Dobrushin uniqueness theorem adapted for lattice systems with a noncompact spin space and proved by Dobrushin and Pechersky. The advantage of this version is that using it one needs to check Dobrushin's contraction condition not for all boundary configurations, but only for those that have spin values in some compact subset of the spin space.

KEY WORDS: Gibbs state; Gibbs random field; uniqueness; specification; configuration; classical gas; spin space.

INTRODUCTION

A series of works on Gibbs models of point field in $\mathbb{R}^d$ was published in the end of 60s and in 70s (for example, see refs. 1, 2, 5, 6, and 8). Nevertheless, it seems that even now there is no satisfactory theory of the Gibbs states of the non-ideal classical gas. Of course, the main problem is still to find examples of phase transitions in these models (see, however, refs. 9 and 10). The uniqueness problem is also of a substantial interest.

In this work we describe the uniqueness region for states of particle systems in $\mathbb{R}^d$ with pair interactions (Sections 1 and 2). Our aim is to show the possibility of applying methods of Dobrushin, which can be found in ref. 3. Namely, we use the general uniqueness theorem from ref. 4 (see

[1] The Dobrushin Mathematical Laboratory, Institute for Problems of Information Transmission, Russian Academy of Sciences, GSP-4 Moscow 101447, Russia.

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Section 3 in this work). There are two Dobrushin results, relevant to this problem (see ref. 3). The first one gives conditions which ensure the existence of at least one Gibbs field. The second Dobrushin theorem gives conditions for the uniqueness of Gibbs field. The latter theorem claims that a set of Gibbs fields corresponding to a specification is either empty or contains a single Gibbs measure provided the Dobrushin uniqueness conditions are satisfied. The theorem in ref. 4 gives similar uniqueness conditions in the above sense as well. The difference is that ref. 4 contains the conditions which work for noncompact spin spaces. The uniqueness conditions in ref. 4 involve both Dobrushin conditions from ref. 3 with some modifications.

The theorem in ref. 4 is proven for lattice models on $\mathbb{Z}^d$. To apply this theorem to continuous models in $\mathbb{R}^d$, we reduce such a continuous model to an equivalent one on $\mathbb{Z}^d$ by appropriate partitions of $\mathbb{R}^d$ into cubes (Section 4). We study the case of pair potentials. However, the method allows to extend the result to multi-body interactions. We restrict our consideration to the case of finite range interactions because the general theorem in ref. 4 is proven for this case only. Figure 1 shows the uniqueness region that we obtain.

In ref. 8, D. Ruelle had obtained the uniqueness results for infinite range superstable potentials. Ruelle's method is to prove that the correlations functions satisfy the Kirkwood–Salsburg equations. Then the uniqueness of the solution of these equations implies the uniqueness of the Gibbs measure. We do not know relations between the conditions we require (Section 1) and superstability. Moreover we do not know if the potentials we consider are stable (see ref. 5). Hence we do not know if the grand canonical partition function exists for all finite volumes.

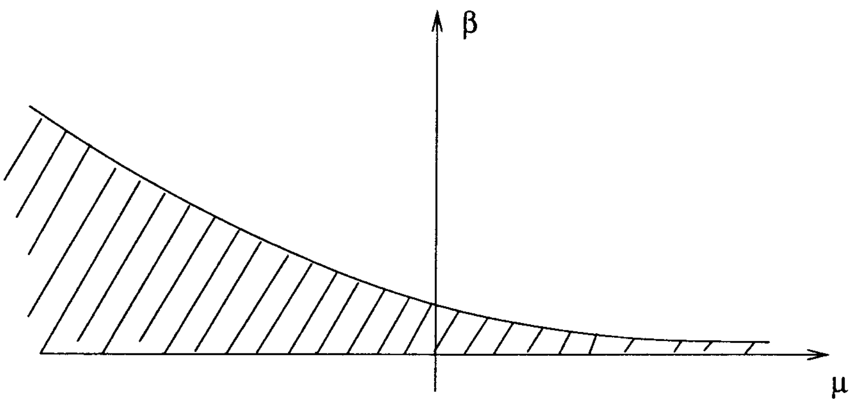

Fig. 1. The uniqueness region (under $\beta_0(K_0)$).

We requires that the potential function is positive near zero which implies the existence of the grand canonical partition function for small volumes in $\mathbb{R}^d$ (see Section 1). That is why we introduce a notion of bounded specifications (Section 1). To prove the main theorem we check the uniqueness conditions from ref. 4 (Sections 5 and 6). If the stability property is fullfield then we can prove the existence of a unique Gibbs measure as well (Section 7).

1. CONTINUOUS MODELS OF THE CLASSICAL GAS

1.1. Configuration Space and Reference Measure

We begin with definitions and notations. *A configuration* of our system is a pair $\sigma=(\mathbf{x}, n)$, where $\mathbf{x}$ is a countable subset in $\mathbb{R}^d$ and n is a map $n\colon \mathbf{x} \to \{1, 2,...\}$ such that for any bounded $V \subset \mathbb{R}^d$

$$\sum_{x \in \mathbf{x} \cap V} n(x) < \infty \tag{1}$$

We denote by X_{cont} the set of all such configurations. We can interpret a pair $\sigma=(\mathbf{x}, n)$ in the following way. The set $\mathbf{x}$ is a set of points from $\mathbb{R}^d$ where particles sit. For every $x \in \mathbf{x}$ the number $n(x)$ is the number of particles sitting at x.

Let V be a Borel set in $\mathbb{R}^d$. Then X^V_{cont} is a set of configurations in V. We use the following notations: $\mathbf{x}_V = \mathbf{x} \cap V$ and $\sigma_V=(\mathbf{x}_V, n_V)$, where $n_V = n|_{\mathbf{x}_V}$, is *the restriction* of $\sigma=(\mathbf{x}, n) \in X_{\text{cont}}$ to V. We say that $\tau=(\mathbf{y}, m) \in X_{\text{cont}}$ is *included* in $\sigma=(\mathbf{x}, n)$ if $\mathbf{y} \subset \mathbf{x}$ and $m(y) \leqslant n(y)$ for $y \in \mathbf{y}$. If $\mathbf{y} \cap \mathbf{x} = \varnothing$, then the *union* $\sigma \cup \tau$ is the configuration $\gamma=(\mathbf{z}, l)$, where $\mathbf{z}=\mathbf{x} \cup \mathbf{y}$ and for $z \in \mathbf{x} \cup \mathbf{y}$

$$l(z)=\begin{cases} n(z), & \text{if } \ z \in \mathbf{x} \\ m(z), & \text{if } \ z \in \mathbf{y} \end{cases}$$

The empty configuration θ is the configuration $\theta=(\varnothing, 0)$, i.e. $\mathbf{x}=\varnothing$ and $n \equiv 0$. We define $\sigma \cap \tau=(\mathbf{w}, k)$, where $\mathbf{w}=\mathbf{x} \cap \mathbf{y}$ and $k(w)=\min\{n(w), m(w)\}$ for $w \in \mathbf{w}$. If $\mathbf{x} \cap \mathbf{y}=\varnothing$ we write $\sigma \cap \tau=\theta$.

Let V be a bounded Borel subset in $\mathbb{R}^d$ and $\sigma=(\mathbf{x}, n) \in X_{\text{cont}}$. The sum $\sum_{x \in \mathbf{x}_V} n(x)$ is finite (see (1)) and we denote this sum by $|\sigma_V|$. We introduce a σ-algebra $\mathfrak{B}$ on the configuration space X_{cont} generated by sets of the type

$$\{\sigma \in X_{\text{cont}} : |\sigma_V| = n\} \tag{2}$$

for some $n \in \mathbb{Z}_+$ and $V \subset \mathbb{R}^d$ (cylindrical sets). The *reference measure* ν for these sets is defined by the formula

$$\nu\{\sigma \in X_{\text{cont}} : |\sigma_V| = n\} = \frac{\lambda(V)^n}{n!}$$

where $\lambda(V)$ is the Lebesgue measure of the set V in $\mathbb{R}^d$. We can extend the measure ν to the whole σ-algebra $\mathfrak{B}$ using the equality

$$\nu\{\mathscr{A}_{n_1}^{(1)} \cap \mathscr{A}_{n_2}^{(2)}\} = \nu\{\mathscr{A}_{n_1}^{(1)}\}\ \nu\{\mathscr{A}_{n_2}^{(2)}\}$$

where

$$\mathscr{A}_{n_i}^{(i)} = \{\sigma \in X_{\text{cont}} : |\sigma_{V_i}| = n_i\}, \qquad i = 1, 2, \quad V_1 \cap V_2 = \varnothing$$

1.2. Conditional Energy

In the next subsection we introduce a Gibbs reconstruction of the measure ν given by a *pair potential* $\hat{\Phi}(x, y)$: $\mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$. Our assumptions on this function are following.

(i) The potential $\hat{\Phi}(x, y)$ is of finite range: there exists a constant $D > 0$ such that for every $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$ with $|x - y| > D$

$$\hat{\Phi}(x, y) = 0$$

(ii) Symmetry: $\hat{\Phi}(x, y) = \hat{\Phi}(y, x)$ for all $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$

(iii) Translation invariance: for any $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$ and any $z \in \mathbb{R}^d$

$$\hat{\Phi}(x + z, y + z) = \hat{\Phi}(x, y)$$

Therefore we can introduce the function $\Phi(x)$, $x \in \mathbb{R}^d$, by the equality

$$\Phi(x - y) = \hat{\Phi}(x, y)$$

(iv) There exists a constant $M > 0$ such that for any $x \in \mathbb{R}^d$

$$\Phi(x) > -M$$

(v) There exist constants $\delta_0 > 0$ and $A > 0$ such that

$$A > 8M\left(3 + \frac{2\sqrt{d}\,D}{\delta_0}\right)^d$$

and

$$\Phi(x) \geqslant A \qquad \text{if} \quad |x| \leqslant \delta_0$$

This condition means that the repulsive part of the potential is greater than the attractive one. This can be seen from the inequality $A(\delta_0)^d > CMD^d$ with $C > (3 + 2\sqrt{d})^d$ which implies (v). In fact this inequality is close to (v). Note that if $|\Phi(x)| \uparrow \infty$ as $|x| \to 0$ and $\int_{|x| < \varepsilon} \Phi(x)\, dx = \infty$ for $\varepsilon > 0$ then (v) is satisfied.

(vi) $\sup_{|x| > \delta_0} |\Phi(x)| < \infty$

We can assume that the constant M in (iv) is such that

$$\sup_{|x| > \delta_0} |\Phi(x)| < M$$

(vii) The function $\Phi(x)$ is finite for all x except may be $x = 0$.

Now we define the *energy* of a finite configuration $\sigma = (\mathbf{x}, n)$ by the formula

$$H(\sigma) = \sum_{\substack{x,\, y \in \mathbf{x} \\ x \neq y}} n(x)\, n(y)\, \Phi(x - y) + \Phi(0) \sum_{x \in \mathbf{x}} \binom{n(x)}{2} \tag{3}$$

where $\binom{k}{2} = k(k-1)/2$. If $n(x) \equiv 1$ and $|\sigma| \geqslant 2$ then

$$H(\sigma) = \sum_{\substack{x,\, y \in \mathbf{x} \\ x \neq y}} \Phi(x - y) \tag{4}$$

For the case $\Phi(0) = \infty$ it is clear that $H(\sigma) = \infty$, if $n(x) > 1$ at least for one $x \in \mathbf{x}$. To simplify the notation we write

$$H(\sigma) = \sum_{x,\, y \in \sigma} \Phi(x - y)$$

instead of (3). We also use the notation $\sigma \subset V$ which means that $\mathbf{x} \subset V$, where $\sigma = (\mathbf{x}, n)$. If σ and τ are finite configurations such that $\sigma \cap \tau = \varnothing$ then their interaction energy $F(\sigma, \tau)$ is defined as

$$F(\sigma, \tau) = H(\sigma \cup \tau) - H(\tau) - H(\sigma) = \sum_{\substack{x \in \sigma \\ y \in \tau}} \Phi(x - y) \tag{5}$$

where the sum in the right hand side is a simplified notation for $\sum_{x \in \sigma, y \in \tau} n(x)\, n(y)\, \Phi(x-y)$.

Let V be bounded Borel set and $\sigma \in X^V_{\text{cont}}$, $\tau \in X^{Vc}_{\text{cont}}$, $V^c = \mathbb{R}^d \setminus V$. By the finite range property of the potential the configuration σ interacts only with the finite part of $\tau = (\mathbf{y}, m)$ situated in the set

$$U = \{ y \in V^c : \inf_{x \in V} |x-y| \leqslant D \} \tag{6}$$

Therefore we have $F(\sigma, \tau) = F(\sigma, \tau_U)$.

The value

$$H(\sigma \mid \tau) = H(\sigma) + F(\sigma, \tau)$$

is called *a conditional Hamiltonian* (under condition τ).

1.3. Specifications

A *Gibbs specification* $\{P_{V,\tau}, V \subset \mathbb{R}^d, \tau \in X^{Vc}_{\text{cont}}\}$ is the following family of Gibbs reconstruction of the measure ν in finite volumes V given by the conditional energy $H(\sigma \mid \tau)$, inverse temperature $\beta \in \mathbb{R}_+$ and chemical potential $\mu \in \mathbb{R}$. Let $\sigma \subset V$, $\tau \subset V^c$, then $P_{V,\tau}$ has the following density $p_{V,\tau}(\sigma)$ with respect to the measure ν:

$$p_{V,\tau}(\sigma) = \frac{\exp\{-\beta H(\sigma \mid \tau) + \mu\, |\sigma|\}}{\int_{X^V_{\text{cont}}} \exp\{-\beta H(\sigma \mid \tau) + \mu\, |\sigma|\}\, \nu(d\sigma)} \tag{7}$$

It is clear that this distribution is well-defined if the integral in the denominator is finite. The well-known stability condition: there exists a constant $R > 0$ such that for any σ with $|\sigma| < \infty$

$$H(\sigma) \geqslant -R\, |\sigma| \tag{8}$$

provides us with this property (see ref. 5). The integral in (7) is called the partition function.

It is not clear if the conditions (i)–(vii) are sufficient for (8). Nevertheless we consider the potential functions Φ satisfying the properties (i)–(vii) only and we avoid the correctness problem considering the densities (7) for small volumes V only. Then the denominator of (7) is finite as follows from two simple lemmas below.

Lemma 1. Let a volume V be such that $\operatorname{diam}(V) < \delta_0$. Then for every $\sigma \subset V$ and $\tau \subset V^c$

$$H(\sigma \mid \tau) > \frac{A}{4} |\sigma|^2 - M |\sigma| \cdot |\tau_U| \tag{9}$$

where U is defined in (6).

Proof. We rewrite the energy in the form

$$H(\sigma \mid \tau) = H(\sigma) + F(\sigma, \tau) \tag{10}$$

(see (3) and (5)). Since $|\sigma| < \infty$

$$H(\sigma) = \sum_{x_1, x_2 \in \sigma} \Phi(x_1 - x_2) \geqslant A \binom{|\sigma|}{2} > A \frac{|\sigma|^2}{4}$$

By our assumptions $\Phi(x, y) \geqslant -M$ (see (iv)) hence we have

$$F(\sigma, \tau) = \sum_{\substack{x \in \sigma \\ y \in \tau}} \Phi(x - y) > -M |\sigma| |\tau_U| \quad \blacksquare$$

Having (9) we obtain

Lemma 2. For V and τ with the same properties as in Lemma 1 the partition function

$$Z_{V, \tau} = \int_{X^V_{\text{cont}}} e^{-\beta H(\sigma \mid \tau) + \mu |\sigma|} \nu(d\sigma)$$

is finite.

Proof. This integral can be written in the form

$$Z_{V, \tau} \equiv \int_{X^V_{\text{cont}}} e^{-\beta H(\sigma \mid \tau) + \mu |\sigma|} \nu(d\sigma) = 1 + \sum_{N=1}^{\infty} e^{\mu N} \int_{X^V_{\text{cont}}} e^{-\beta H(\sigma \mid \tau)} \nu(d\sigma)$$

where $X^N_{\text{cont}} = \{\sigma \in X^V_{\text{cont}} : |\sigma_V| = N\}$. The proof follows from (9). $\blacksquare$

We introduce a *bounded* specification $\{P_{V, \tau} : \operatorname{diam}(V) < \delta_0,\ \tau \in X^{V^c}_{\text{cont}}\}$. Further we shall omit the word "bounded" except for the cases which lead to an ambiguity.

A random point field $\xi = \{\xi_V, V \subset \mathbb{R}^d\}$ *corresponds to the specification* $\{P_{V,\tau}\}$ if

$$\Pr(\xi_V \in \mathscr{A} \mid \xi_{V^c} = \tau) = P_{V,\tau}(\mathscr{A}) \tag{11}$$

for any bounded Borel set $V \subset \mathbb{R}^d$ and for any $\mathscr{A} \subseteq X^V_{\text{cont}}$.

A random point field $\xi = \{\xi_V, V \subset \mathbb{R}^d\}$ *corresponds to the bounded specification* $\{P_{V,\tau} : \text{diam}(V) < \delta_0\}$ if

$$\Pr(\xi_V \in \mathscr{A} \mid \xi_{V^c} = \tau) = P_{V,\tau}(\mathscr{A}) \tag{12}$$

for any bounded Borel set $V \subset \mathbb{R}^d$ with $\text{diam}(V) < \delta_0$ and for any $\mathscr{A} \subseteq X^V_{\text{cont}}$.

2. MAIN RESULT

We study uniqueness conditions for fields ξ corresponding to the specification $\{P_{V,\tau}\}$. Let $\chi > 0$. We introduce a class $\mathfrak{A}_\chi$ of random fields ξ corresponding to a given specification $\{P_{V,\tau}\}$ such that

$$\mathsf{E} e^{\chi\, |\xi_V|} < \infty \tag{13}$$

for every bounded $V \subseteq \mathbb{R}^d$ such that $\text{diam}(V) \leqslant \delta_0$.

Theorem 3. Let the specification $\{P_{V,\tau}\}$ be defined by the potential function Φ and the conditions (i)–(vii) are satisfied for Φ. Then for any $\chi > 0$ and any $\mu \in \mathbb{R}$ there exists $\beta(\chi)$ such that for $\beta \leqslant \beta(\chi)$

$$|\mathfrak{A}_\chi| \leqslant 1$$

The symbol $|\mathfrak{A}_\chi|$ means the number of elements in the set $\mathfrak{A}_\chi$.

The proof of Theorem 3 is based on the uniqueness theorem from ref. 4 for lattice Gibbs models. In Section 4 we will show how to construct a lattice model related to our system. This lattice model is equivalent with respect of the uniqueness to the original continuous model introduced in Section 1. For reader's convenience we in the next section quote the theorem from ref. 4.

3. UNIQUENESS THEOREM ON $\mathbb{Z}^D$

Let S be a complete separable metric space, and $V \subseteq \mathbb{Z}^d$. *A configuration* $\bar{\mathbf{x}}$ on V is a map $\bar{\mathbf{x}}$: $V \to S$, S^V is the set of all configurations on V. Let

$\{P_{t,\bar{\mathbf{x}}}\}$ be the family of probability measures on S (*a specification*) indexed by parameters $t \in \mathbb{Z}^d$ and $\bar{\mathbf{x}} \in S^{\mathbb{Z}^d \setminus \{t\}}$.

A random field $\xi(t)$, $t \in \mathbb{Z}^d$, taking values in S *corresponds* to the specification $\{P_{t,\bar{\mathbf{x}}}\}$ if for every $t \in \mathbb{Z}^d$ and every Borel set $\mathscr{A} \subseteq S$

$$\Pr\{\xi(t) \in \mathscr{A} \mid \xi(u) = \bar{\mathbf{x}}(u), u \neq t\} = P_{t,\bar{\mathbf{x}}}(\mathscr{A})$$

Denote by

$$\partial t = \{u \in \mathbb{Z}^d : |u - t| \leqslant r\}$$

the vicinity of the point $t \in \mathbb{Z}^d$, where $|\cdot|$ is some norm in $\mathbb{Z}^d$ and $r > 0$.

We consider finite-range interactions. This means that there exists $r > 0$ such that for every pair of configurations $\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2 \in S^{\mathbb{Z}^d \setminus \{t\}}$ with $\bar{\mathbf{x}}_1(u) = \bar{\mathbf{x}}_2(u)$ for $u \in \partial t$, the following equality is valid

$$P_{t,\bar{\mathbf{x}}_1} = P_{t,\bar{\mathbf{x}}_2}$$

Let a be the number of elements in ∂t and Z_0 be a subgroup of $\mathbb{Z}^d$ such that for any $u, v \in Z_0$ the inequality $|u - v| > r$ is valid. We set

$$b = \min_{Z_0} |\mathbb{Z}^d / Z_0|$$

where $|\mathbb{Z}^d / Z_0|$ is the number of points in the quotient group $\mathbb{Z}^d / Z_0$.

The *compact function* is anon-negative continuous function $h: S \to \mathbb{R}_+$ such that for any $h_0 \in \mathbb{R}_+$ the set

$$K(h_0) = \{x \in S = h(x) \leqslant h_0\}$$

is compact in S.

Now we formulate a number of conditions on the specification $\{P_{t,\bar{\mathbf{x}}}\}$ required by the uniqueness theorem of ref. 4.

Compactness Condition. There exist a compact function $h(x)$ on S and constants $C \geqslant 0$ and $c_u \geqslant 0$, $u \in \partial 0$, such that

C1. $\sum_{u \in \partial 0} c_u < 1/a^b$;

C2. For any $t \in \mathbb{Z}^d$ and for any configuration $\bar{\mathbf{x}} \in S^{\mathbb{Z}^d \setminus \{t\}}$

$$\int_S h(x)\, P_{t,\bar{\mathbf{x}}}(dx) \leqslant C + \sum_{u \in \partial 0} c_u h(\bar{\mathbf{x}}(t+u))$$

Given $\rho < 1$ denote by $\Theta(h, C, \rho)$ the class of all specifications satisfying this compactness condition with $a^b \sum c_u \leqslant \rho$.

Contraction Condition. There exist a compact function $h(x)$ on S and constants $K \geqslant 0$ and $k_u \geqslant 0$, $u \in \partial 0$, such that

D1. $\sum_{u \in \partial 0} k_u < 1$;

D2. For any $t \in \mathbb{Z}^d$ and for any pair of configurations $\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2 \in S^{\mathbb{Z}^d \setminus \{t\}}$ such that

$$\max\{h(\bar{\mathbf{x}}_i(u)) : u \in \partial t, i = 1, 2\} \leqslant K$$

the following inequality is valid

$$V(P_{t, \bar{\mathbf{x}}_1}, P_{t, \bar{\mathbf{x}}_2}) \leqslant \sum_{u \in \partial 0} k_u \delta(\bar{\mathbf{x}}_1(t+u), \bar{\mathbf{x}}_2(t+u))$$

where $V(P, Q)$ is the variation distance between two measures P and Q, and

$$\delta(\mathbf{x}, \mathbf{y}) = \begin{cases} 0, & \mathbf{x} = \mathbf{y} \\ 1, & \mathbf{x} \neq \mathbf{y} \end{cases}$$

We denote by $\Delta(h, K, \alpha)$ the class of all specifications satisfying this contraction condition with $\sum k_u \leqslant \alpha < 1$.

Let $\Gamma \in \Theta(h, C, \rho) \cap \Delta(h, K, \alpha)$ and let $\mathfrak{R}_h$ be the class of random fields ξ on $\mathbb{Z}^d$ with values in S corresponding to the specification Γ and satisfying the condition

$$\sup_{t \in \mathbb{Z}^d} \mathsf{E} h(\xi(t)) < \infty$$

Theorem 4 (ref. 4). Let $\alpha, \rho \in (0, 1)$, $C > 0$ and h be a compact function. Then there exists a value $\hat{K} = \hat{K}(C, a, b, \alpha, \rho)$ such that for any $\Gamma \in \Theta(h, C, \rho) \cap \Delta(h, K, \alpha)$ with $K \geqslant \hat{K}$ the set $\mathfrak{R}_h$ is either empty or consists of a single-element.

The value $\hat{K}$ indeed depends on C, a, b, α, and ρ. This can be seen from the proof of the main theorem in ref. 4 (see Lemma 4 in ref. 4).

Note that the "compactness condition" (see the condition C1) is stronger than the original Dobrushin's condition from ref. 3. Introducing condition (C1)–(C2) one gains that the "contraction condition" (the condition D2) need to be checked for the boundary configurations belonging to a compact only. This is the main advantage of the combination of both conditions.

In our applications of the above theorem we prove that for every $\mu \in \mathbb{R}$ and every $K > 1$ the contraction condition is satisfied if β is small. Therefore we can apply the theorem even if no estimates on $\hat{K}$ are known.

It is possible to give a lower estimate of $\hat{K}$. We will do this elsewhere.

Note that we need only one-point specification $\{P_{t, \bar{\mathbf{x}}}\}$ in the theorem.

4. FROM CONTINUOUS TO LATTICE MODEL

In this section we explain how to construct a lattice model equivalent to the continuous model of Section 1. Then one can apply the uniqueness theorem from the previous section.

4.1. Spin Space

Partition the space $\mathbb{R}^d$ into equal cubes with edges parallel to coordinate planes and index these cubes in natural way by the points of $\mathbb{Z}^d$. The cube G_0 containing the origin $0 \in \mathbb{R}^d$ is

$$G_0 = \{x = (x^{(1)},..., x^{(d)}) \in \mathbb{R}^d : -g \leqslant x^{(i)} < g, i = 1,..., d\} \tag{14}$$

So we have

$$\mathbb{R}^d = \bigcup_{t \in \mathbb{Z}^d} G_t, \; G_t = G_0 + 2gt$$

We will choose the parameter g such that $\operatorname{diam}(G_t) \leqslant \delta_0$. This implies that there is only a repulsive interactions inside any of G_t (see Section 1, the condition (v)).

We take the set $S = X_{\text{cont}}^{\bar{G}_0}$ ($\bar{G}_0$ is the closure of G_0) as a *spin space*. We provide this spin space with the following metric. Let $\sigma = (\mathbf{x}, n)$ and $\tau = (\mathbf{y}, m)$ be particle configurations in $S = X_{\text{cont}}^{\bar{G}_0}$ and let $\mathbf{x}' = (x_1,..., x_{|\sigma|})$ be a sequence of points from the set $\mathbf{x}$ such that each point $x_i \in \mathbf{x}$ is encountered $n(x_i)$ times in $\mathbf{x}'$. Similarly, introduce $\mathbf{y}' = (y_1,..., y_{|\tau|})$. The distance $r(\sigma, \tau)$ between these configurations $\sigma = (\mathbf{x}, n)$ and $\tau = (\mathbf{y}, m)$ is defined as follows

$$r(\sigma, \tau) = \begin{cases} \dfrac{1}{2g\sqrt{d}\,|\sigma|} \min\limits_{\pi} \sum\limits_{i=1}^{|\sigma|} |x_i - y_{\pi(i)}|, & \text{if } |\sigma| = |\tau| \\ 1, & \text{otherwise} \end{cases} \tag{15}$$

In this formula the minimum is taken over the set of all permutations π of the set $\{1,..., |\sigma|\}$. The metric space $\langle S, r \rangle$ is complete.

4.2. Lattice Configuration Space X_{lat}

Define a *lattice configuration* $\bar{\sigma}$ as a map $\bar{\sigma}: \mathbb{Z}^d \to S$. By $X_{\text{lat}} \equiv S^{\mathbb{Z}^d}$ we denote the set of all lattice configurations. We can interpret every configuration $\bar{\sigma}$ as a pair $\bar{\sigma} = (\bar{\mathbf{x}}(t), \bar{n}_t)$, where $\bar{\mathbf{x}}(t)$ is a finite subset in $\bar{G}_0$ and $\bar{n}_t$ is a function $n_t: \bar{\mathbf{x}}(t) \to \{1, 2,...\}$.

Note that if $t, s \in \mathbb{Z}^d$ are nearest neighbors then it is possible that $\bar{\mathbf{x}}(t) \cap \bar{\mathbf{x}}(s) \neq \varnothing$ since $\bar{G}_t \cap \bar{G}_s \neq \varnothing$.

4.3. Correspondence Between Continuous and Lattice Configuration Spaces

A lattice configuration $\bar{\sigma} = (\bar{\mathbf{x}}(t), \bar{n}_t) \in X_{\text{lat}}$ corresponds to the continuous configuration $\sigma = (\mathbf{x}, n) \in X_{\text{cont}}$ if

$$\bar{\mathbf{x}}(t) = \mathbf{x} \cap G_t - 2gt$$

and for $x \in \bar{\mathbf{x}}(t)$

$$\bar{n}_t(x) = n(x + 2gt)$$

where $G_t = G_0 + 2gt$. Denote this correspondence by $\mathbf{T}: X_{\text{cont}} \to X_{\text{lat}}$. Hence $\bar{\sigma} = \mathbf{T}\sigma = (\bar{\mathbf{x}}(t), \bar{n}_t)$ with $\bar{\mathbf{x}}(t) \subset G_0 \subset \bar{G}_0$. Therefore the range $\mathbf{T}(X_{\text{cont}})$ is strictly less than X_{lat}:

$$\mathbf{T}(X_{\text{cont}}) \subset X_{\text{lat}}$$

and $\mathbf{T}$ is an injective map.

The inverse map $\mathbf{T}^{-1}$ acts a follows. If $\bar{\sigma} = (\bar{\mathbf{x}}(t), \bar{n}_t) \in \mathbf{T}(X_{\text{cont}})$ then $\sigma = (\mathbf{x}, n) = \mathbf{T}^{-1}\bar{\sigma}$ is defined by

$$\mathbf{x} = \bigcup_{t \in \mathbb{Z}^d} (\bar{\mathbf{x}}(t) + 2gt) \tag{16}$$

Note that the sets $\bar{\mathbf{x}}(t)$ and $\bar{\mathbf{x}}(s)$ do not intersect for $s \neq t$ and for any $x \in \mathbf{x}$ there exists only one point $t \in \mathbb{Z}^d$ such that $x \in \bar{\mathbf{x}}(t) + 2gt$. This allows us to define n:

$$n(x) = \bar{n}_t(x - 2gt)$$

The map $\mathbf{T}$ is a measurable imbedding. Hence every measure on X_{cont} induce a measure on X_{lat} and thereby, each specification on X_{cont} generates a specification for the lattice model. The inverse map $\mathbf{T}^{-1}$ can be extended from $\mathbf{T}(X_{\rm cont})$ to the whole $X_{\rm lat}$. If $\bar\sigma=(\bar{\mathbf{x}}(t),\bar n(t))\in X_{\rm lat}\setminus\mathbf{T}(X_{\rm cont})$ then there exist sites $t\in\mathbb{Z}^d$ such that some particles from $\bar{\mathbf{x}}(t)$ are situated at a point $x\in\bar{\mathbf{x}}(t)$ which is in $\bar G_0\setminus G_0$. Nevertheless defining $\mathbf{T}^{-1}$ for this case we can use (16). To define $n(\cdot)$ from $\bar n_t(\cdot)$ we put for $x\in\mathbf{x}$

$$n(x)=\sum_{t:\, x\in\bar{\mathbf{x}}(t)+2gt}\bar n_t(x-2gt)$$

4.4. Energy of a Lattice Configuration

We find the Hamiltonian of the lattice model using the Hamiltonian of the continuous model. Let $V\subseteq\mathbb{Z}^d$ be a finite volume and $\bar\tau$ be a configuration out of V.

Then the conditional energy $\bar H(\bar\sigma\mid\bar\tau)$ of a configuration $\bar\sigma\in X_{\rm lat}$ on V under the condition $\bar\tau$ is equal

$$\bar H(\bar\sigma\mid\bar\tau)=H((\mathbf{T}^{-1}\bar\sigma)_{\bigcup_{t\in V}G_t}\mid(\mathbf{T}^{-1}\bar\tau)_{\bigcup_{t\in V^c}G_t}$$

where $V^c=\mathbb{Z}^d\setminus V$, $(\mathbf{T}^{-1}\bar\sigma)_W$ is the restriction of the configuration $\mathbf{T}^{-1}\bar\sigma\in X_{\rm cont}$ to the set $W\in\mathbb{R}^d$, and $(\mathbf{T}^{-1}\bar\tau)_W$ is defined similarly.

Let now V, $W\subset\mathbb{Z}^d$, $V\cap W=\varnothing$, be finite sets, $\bar\sigma$ be a configuration on V and $\bar\tau$ be a configuration on W. Then the interaction energy between $\bar\sigma$ and $\bar\tau$ is equal to

$$\bar F(\bar\sigma,\bar\tau)=F((\mathbf{T}^{-1}\bar\sigma)_{\bigcup_{t\in V}G_t},(\mathbf{T}^{-1}\bar\tau)_{\bigcup_{t\in W}G_t})$$

If $u, v\in\mathbb{Z}^d$ and $\operatorname{dist}(\bar G_u,\bar G_v)>D$ then for any configuration $\bar\sigma\in X_{\rm lat}$

$$\bar F(\bar\sigma(u),\bar\sigma(v))=0$$

Hence it is naturally to define a boundary of a point $t\in\mathbb{Z}^d$ by the equality

$$\partial t=\{u\in\mathbb{Z}^d:\operatorname{dist}(\bar G_u,\bar G_t)\leqslant D\}$$

In the sequel we assume that the quantities a and b in Theorem 4 are related to this definition of the boundary.

4.5. Reference Measure and Specifications in Lattice Model

Defining lattice specifications corresponding to the bounded specifications $\{P_{V,\tau}:\operatorname{diam}(V)<\delta_0\}$ we take the partitions on $\mathbb{R}^d$ by cubes $\{G_t\}$

such that $g < \delta_0/2\sqrt{d}$ (see (v) and (14)). Then it follows from Lemma 2 that

$$Z_{\bar{G}_{0,\tau}} = \int_{X_{\text{cont}}^{\bar{G}_0}} \exp\{-\beta H(\sigma \mid \tau) + \mu\, |\sigma|\}\, \nu(d\sigma) < \infty$$

for any $\tau \in X_{\text{cont}}^{\bar{G}_0^c}$, $\mu \in \mathbb{R}$ and $\beta > 0$. Hence we can properly define the single-point specification for the lattice counterpart of the continuous model. Namely, the density $p_{t,\bar{\tau}}$ of $P_{t,\bar{\tau}}$ is given by

$$p_{t,\bar{\tau}}(\bar{\sigma}) = \frac{\exp\{-\beta \bar{H}(\bar{\sigma} \mid \bar{\tau}) + \mu\, |\bar{\sigma}|\}}{Z_{t,\bar{\tau}}}$$

where

$$Z_{t,\bar{\tau}} = \int_S \exp\{-\beta \bar{H}(\bar{\sigma} \mid \bar{\tau}) + \mu\, |\bar{\sigma}|\}\, \nu(d\bar{\sigma}) = Z_{\bar{G}_{t,\tau}}$$

and $\bar{\sigma} = \mathbf{T}\sigma$, $\bar{\tau} = \mathbf{T}\tau$.

Note that $P_{t,\bar{\tau}}$ and $Z_{t,\bar{\tau}}$ do not depend on t.

In the next sections we check the conditions of Theorem 4 for the lattice specification $\{P_{t,\bar{\tau}}\}$. It is clear from the construction in this section that the uniqueness of the lattice model implies the uniqueness of the corresponding continuous model. It is not difficult to show that the uniqueness properties of both models are equivalent.

In the sequel when working with the lattice model we will omit the bars over the letters H, σ, τ and etc.

5. COMPACTNESS CONDITION

In this section we prove that the compactness condition is satisfied with the compact function $h(\sigma) = e^{\chi\, |\sigma|}$ on S.

Lemma 5. There are constants $C > 0$ and $0 < c < 1/a^{b+1}$ independent on β such that for every $t \in \mathbb{Z}^d$ and any boundary condition $\tau \in X_{\text{lat}}^{\mathbb{Z}^d \setminus \{t\}}$

$$\mathsf{E}_\tau h = \int_S h(\sigma)\, P_{t,\tau}(d\sigma) \leqslant C + \frac{c}{a^{b+1}} \sum_{u \in \partial t} h(\tau(u))$$

Moreover, we can take

$$C = \exp\{e^{\chi+\mu}(\delta_0)^d\} + \frac{a^b}{2}$$

We would like to stress that the right hand side in the lemma of the above inequality is independent on β.

Proof. We use the representation

$$\mathsf{E}_\tau h = \sum_{N \leqslant N_\tau} \int_{S^N} h(\sigma)\, P_{t,\tau}(d\sigma) + \sum_{N > N_\tau} \int_{S^N} h(\sigma)\, P_{t,\tau}(d\sigma) \tag{17}$$

where

$$S^N = \{\sigma \in S : |\sigma| = N\} \qquad \text{and} \qquad S = \bigcup_{N=0}^{\infty} S^N$$

The real number N_τ will be chosen later. First we estimate the second term on the right hand side of (17). Since the partition function $Z_{t,\tau}$ is greater than 1 we obtain

$$\begin{aligned} I_{N_\tau} \equiv \sum_{N > N_\tau} \int_{S^N} h(\sigma)\, P_{t,\tau}(d\sigma) &\leqslant \sum_{N > N_\tau} e^{(\mu+\chi) N} \int_{S^N} e^{-\beta H(\sigma \mid \tau)} \nu(d\sigma) \\ &\leqslant \sum_{N > N_\tau} e^{(\mu+\chi) N} \exp\left\{\beta N \left[M\, |\tau| - \frac{A}{4} N \right]\right\} \frac{(\delta_0^d)^N}{N!} \end{aligned}$$

where $|\tau| = \sum_{u \in \partial t} |\tau(u)|$. In the last inequality we used (9). Choosing $N_\tau = (4M\, |\tau|)/A$ we have for all $N > N_\tau$

$$M\, |\tau| \leqslant \frac{A}{4} N$$

This implies

$$I_{N_\tau} \leqslant \sum_{N > N_\tau} e^{(\mu+\chi) N} \frac{(\delta_0^d)^N}{N!} \leqslant \exp\{e^{\chi+\mu} \delta_0^d\} - 1 \tag{18}$$

Estimating the first term in (17) first we observe that

$$\begin{aligned} J_{N_\tau} &\equiv \sum_{N \leqslant N_\tau} \int_{S^N} h(\sigma)\, P_{t,\tau}(d\sigma) \\ &= \sum_{N \leqslant N_\tau} \hat{h}(N)\, P_{t,\tau}(S^N) \leqslant \hat{h}(N_\tau) = \exp\{\chi N_\tau\} \end{aligned}$$

where $\hat{h}(k) = e^{\chi k}$. Hence

$$J_{N_\tau} \leqslant e^{\chi(4M/A)\,|\tau|} = \exp\left\{\chi \frac{4M}{A} \sum_{u \in \partial t} |\tau(u)|\right\}$$

Next we use the inequality

$$\prod_{i=1}^{n} x_i \leqslant \frac{1}{n} \sum_{i=1}^{n} x_i^n \tag{19}$$

which holds for any non-negative $x_1,..., x_n$. We obtain

$$J_{N_\tau} \leqslant \frac{1}{a} \sum_{u \in \partial t} e^{\chi(4M/A)\,a\,|\tau(u)|}$$

Recall that $a = |\partial t|$. Let $T > 0$. Using (19) with $n = 2$ we obtain

$$\begin{aligned} J_{N_\tau} &\leqslant \frac{1}{a} \sum_{u \in \partial t} e^{T} e^{\chi(4M/A)\,a\,|\tau(u)| - T} \\ &\leqslant \frac{1}{2a} \sum_{u \in \partial t} e^{2T} + \frac{1}{2a} \sum_{u \in \partial t} e^{-2T} \hat{h}\left(\frac{8M}{A}\,a\,|\tau(u)|\right) \end{aligned}$$

By the convexity of the function $\hat{h}$

$$\begin{aligned} J_{N_\tau} &\leqslant \frac{e^{2T}}{2} + \frac{1}{2a} \sum_{u \in \partial t} e^{-2T} \hat{h}\left(\left[1 - \frac{8M}{A}\,a\right] \cdot 0 + \frac{8M}{A}\,a\,|\tau(u)|\right) \\ &\leqslant \frac{e^{2T}}{2} + \frac{1}{2a} \sum_{u \in \partial t} e^{-2T} \left\{\left[1 - \frac{8M}{A}\,a\right] h(\theta) + \frac{8M}{A}\,a h(\tau(u))\right\} \end{aligned}$$

By simple evaluations, $a \leqslant (2\lceil D/2g \rceil + 1)^d \leqslant (2\sqrt{d}\,D/\delta_0 + 3)^d)$, where $\lceil z \rceil$ is the least integer greater than or equal to z. Therefore assumption (v) implies $1 - 8M/Aa > 0$, and the last fact as used above in the estimate of J_{N_τ}. Now we have

$$J_{N_\tau} \leqslant \frac{e^{2T}}{2} + \frac{e^{-2T}}{2}\left[1 - \frac{8M}{A}\,a\right] + \sum_{u \in \partial t} c_u h(\tau(u)) \tag{20}$$

where

$$c_u = \frac{4Me^{-2T}}{A} \qquad \text{for any} \quad u \in \partial t \tag{21}$$

We set $T = \frac{1}{2} \ln[(4M/A)\, a^{b+1}(1+\varepsilon)]$ for choosing $\varepsilon > 0$ such that $(4M/A)\, a^{b+1}(1+\varepsilon) \geqslant 1$. Then

$$c = c_u = \frac{1}{a^{b+1}} \frac{1}{1+\varepsilon} \tag{22}$$

It follows from (18) and (20) that

$$\int_S h(\sigma)\, P_{t,\tau}(d\sigma) \leqslant C + \sum_{u \in \partial t} c_u h(\tau(u))$$

where

$$\begin{aligned} C &= \exp\{e^{\chi+\mu}(\delta_0)^d\} + \frac{a^b}{2} \\ &\geqslant \frac{e^{2T}}{2} + \frac{e^{-2T}}{2}\left[1 - \frac{8M}{A}a\right] + \exp\{e^{\chi+\mu}\delta_0^d\} - 1 \quad \blacksquare \end{aligned} \tag{23}$$

6. CONTRACTION CONDITION

Lemma 6. Let $K \geqslant 1$ and $\chi > 0$, and set $K_0 = (1/\chi) \ln K$. For any $\mu \in \mathbb{R}$ there exists $\beta_0(K_0)$ (see Fig. 1) such that for all β: $0 \leqslant \beta < \beta_0(K_0)$ and for any $t \in \mathbb{Z}^d$ and $\tau_1, \tau_2 \in S^{\mathbb{Z}^d \setminus \{t\}}$ such that

$$|\tau_i(u)| < K_0, \qquad i = 1, 2, \quad u \in \partial t$$

the following inequality for the variation distance is valid

$$V(P_{t,\tau_1}, P_{t,\tau_2}) < \sum_{u \in \partial t} k_{u-t}\delta(\tau_1(u), \tau_2(u)) \tag{24}$$

where $k_u = \alpha/a$ for all $u \in \partial 0$ and some $\alpha < 1$.

Proof. Because of the translation invariance of the potential we can perform all our calculations for $t = 0 \in \mathbb{Z}^d$. Therefore we omit t from all subscripts in this proof. Take $\tau_1, \tau_2 \in S^{\mathbb{Z}^d \setminus \{0\}}$ and assume that $\tau_1(u_0) \neq \tau_2(u_0)$ for $u_0 \in \partial 0$ and $\tau_1(u) = \tau_2(u)$ for $u \in \partial 0$, $u \neq u_0$ and $|\tau_2(u_0)| = |\tau_1(u_0)| + 1$. Moreover we assume that for $\tau_1(u_0) = (\mathbf{x}_1, n_1)$ and $\tau_2(u_0) = (\mathbf{x}_2, n_2)$ either

$\mathbf{x}_2 = \mathbf{x}_1 \cup \{y_0\}$, where $y_0 \in \bar{G}_0$, $y_0 \notin \mathbf{x}_1$ and $n_2(y_0) = 1$, or $\mathbf{x}_2 = \mathbf{x}_1$ and for some $y_0 \in \mathbf{x}_2$ $n_2(y_0) = n_1(y_0) + 1$. Consider the variation distance

$$V(P_{\tau_1}, P_{\tau_2}) = \tfrac{1}{2} \int_S |p_{\tau_1}(\sigma) - p_{\tau_2}(\sigma)| \, \nu(d\sigma)$$
$$= \tfrac{1}{2} \sum_{N=0}^{\infty} \int_{S^N} |p_{\tau_1}(\sigma) - p_{\tau_2}(\sigma)| \, \nu(d\sigma)$$

where

$$S = \bigcup_{N=0}^{\infty} S^N, \qquad S^N = \{\sigma : |\sigma| = N\}$$

Thus

$$V(P_{\tau_1}, P_{\tau_2}) \leqslant \frac{1}{2} \sum_{N=0}^{\infty} e^{\mu N} \int_{S^N} \left| \frac{e^{-\beta H(\sigma \mid \tau_1)}}{Z_{\tau_1}} - \frac{e^{-\beta H(\sigma \mid \tau_2)}}{Z_{\tau_2}} \right| \nu(d\sigma)$$

If $N = 0$ then $H(\sigma \mid \tau_i) \equiv H(\theta \mid \tau_i) = 0$, where θ is an empty configuration. We have

$$\left| \frac{1}{Z_{\tau_2}} - \frac{1}{Z_{\tau_1}} \right| = \left| \frac{Z_{\tau_1} - Z_{\tau_2}}{Z_{\tau_1} Z_{\tau_2}} \right| \leqslant |Z_{\tau_2} - Z_{\tau_1}|$$

since for any τ

$$Z_\tau = 1 + \sum_{N=1}^{\infty} e^{\mu N} \int_{S^N} e^{-\beta H(\sigma \mid \tau)} \nu(d\sigma) \geqslant 1$$

Therefore

$$\left| \frac{1}{Z_{\tau_1}} - \frac{1}{Z_{\tau_2}} \right| \leqslant |Z_{\tau_2} - Z_{\tau_1}|$$
$$\leqslant \sum_{N=1}^{\infty} e^{\mu N} \int_{S^N} |e^{-\beta H(\sigma \mid \tau_1)} - e^{-\beta H(\sigma \mid \tau_2)}| \, \nu(d\sigma) \tag{25}$$

For $\sigma \in S^N$ if $N \geqslant 1$ we have

$$\left| \frac{e^{-\beta H(\sigma \mid \tau_1)}}{Z_{\tau_1}} - \frac{e^{-\beta H(\sigma \mid \tau_2)}}{Z_{\tau_2}} \right| \leqslant |Z_{\tau_2} e^{-\beta H(\sigma, \mid \tau_1)} - Z_{\tau_1} e^{-\beta H(\sigma \mid \tau_2)}| \tag{26}$$

It follows from (25) and (26) that

$$
\begin{aligned}
V(P_{\tau_1}, P_{\tau_2})
&\leqslant \sum_{N=1}^{\infty} e^{\mu N} \int_{S^N} |e^{-\beta H(\sigma \mid \tau_1)} - e^{-\beta H(\sigma \mid \tau_2)}| \, v(d\sigma) \\
&\quad + \frac{1}{2} \sum_{N=1}^{\infty} \sum_{N'=1}^{\infty} e^{\mu(N+N')} \int_{S^N} \int_{S^{N'}} \\
&\quad \times |e^{-\beta H(\sigma \mid \tau_1) - \beta H(\sigma' \mid \tau_2)} - e^{-\beta H(\sigma \mid \tau_2) - \beta H(\sigma' \mid \tau_1)}| \, v(d\sigma) \, v(d\sigma')
\end{aligned}
$$

Observe that

$$
H(\sigma \mid \tau_2) = H(\sigma \mid \tau_1) + \sum_{x \in \sigma} \Phi(x - (y_0 + 2gu_0))
$$

Using this and a simple inequality

$$
\begin{aligned}
&|e^{-\beta \sum_{x \in \sigma'} \Phi(x - (y_0 + 2gu_0))} - e^{-\beta \sum_{x \in \sigma} \Phi(x - (y_0 + 2gu_0))}| \\
&\quad \leqslant |1 - e^{-\beta \sum_{x \in \sigma'} \Phi(x - (y_0 + 2gu_0))}| + |1 - e^{-\beta \sum_{x \in \sigma} \Phi(x - (y_0 + 2gu_0))}|
\end{aligned}
$$

we obtain

$$
\begin{aligned}
V(P_{\tau_1}, P_{\tau_2})
&\leqslant \sum_{N=1}^{\infty} e^{\mu N} \int_{S^N} e^{-\beta H(\sigma \mid \tau_1)} \, |1 - e^{-\beta \sum_{x \in \sigma} \Phi(x - (y_2 + 2gu_0))}| \, v(d\sigma) \\
&\quad + \sum_{N=1}^{\infty} \sum_{N'=1}^{\infty} e^{\mu(N+N')} \int_{S^N} e^{-\beta H(\sigma \mid \tau_1)} v(d\sigma) \\
&\quad \times \int_{S^{N'}} e^{-\beta H(\sigma' \mid \tau_1)} \, |1 - e^{-\beta \sum_{x \in \sigma'} \Phi(-(y_0 + 2gu_0))}| \, v(d\sigma') \\
&\leqslant Y_{\tau_1}(y_0) \, Z_{\tau_1}
\end{aligned}
\tag{27}
$$

where

$$
\begin{aligned}
Y_{\tau_1}(y_0) = \sum_{N'=1}^{\infty} \int_{S^{N'}} & e^{-\beta H(\sigma' \mid \tau_1) + \mu N'} \\
& \times |1 - e^{-\beta \sum_{x \in \sigma'} \Phi(x - (y_0 + 2gu_0))}| \, v(d\sigma')
\end{aligned}
\tag{28}
$$

It follows from Lemma 1 that

$$Z_{\tau_1} \leqslant \exp\{e^{\mu + \beta M K_0 a}(2g)^d\} \tag{29}$$

Estimating (28) we use Lemma 1 again:

$$\int_{S^N} e^{-\beta H(\sigma' \mid \tau_1) + \mu N} \, |1 - e^{-\beta \sum_{x \in \sigma'} \Phi(x - (y_0 + 2g u_0))}| \, \nu(d\sigma')$$

$$\leqslant \frac{1}{N!} \, e^{\beta M K_0 a N + \mu N} J_N(\beta)$$

where

$$J_N(\beta) = \int_{\bar{G}_0^N} |1 - e^{-\beta \sum_{i=1}^N \Phi(x_i - (y_0 + 2g u_0))}| \, dx_1 \cdots dx_N \tag{30}$$

We divide the cube $\bar{G}_0$ into two parts

$$\bar{G}_0 = \bar{G}_0^{(1)} \cup \bar{G}_0^{(2)}$$

where

$$\bar{G}_0^{(1)} = \{x : x \in \bar{G}_0, \, |x - (y_0 + 2g u_0)| > \delta_0\}$$

and

$$\bar{G}_0^{(2)} = \{x : x \in \bar{G}_0, \, |x - (y_0 + 2g u_0)| \leqslant \delta_0\}$$

Then

$$J_N(\beta) = \sum_{\alpha} J_N^{\alpha}(\beta) \tag{31}$$

where the sum is taken over multi-indices $\alpha = (\alpha_1, \ldots, \alpha_N)$, α_i is equal to either 1 or 2, and $J_N^{\alpha}(\beta)$ is the integral of the same function as in (30) taken over the region $\bar{G}_0^{(\alpha)} = \bar{G}_0^{(\alpha_1)} \times \bar{G}_0^{(\alpha_2)} \times \cdots \times \bar{G}_0^{(\alpha_N)}$.

Next we estimate every term in (31). For $\alpha = (\alpha_1, \ldots, \alpha_N)$ let $I^1_{N, \alpha} = \{i : 1 \leqslant i \leqslant N, \, \alpha_i = 1\}$ and $I^2_{N, \alpha} = \{1, \ldots, N\} \setminus I^1_{N, \alpha}$. We apply the inequality

$$|1 - C e^{-\sum_{k=1}^N a_k}| \leqslant \sum_{k=1}^N |1 - e^{-a_k}| + |1 - C|$$

valid for any $C \geqslant 0$ and $a_k \geqslant 0$, $k = 1,..., N$. Since $\Phi(x-(y_0+2gu_0)) > 0$ for $x \in G_0^{(2)}$ we have

$$|1 - e^{-\beta \sum_{i=1}^{N} \Phi(x_i - (y_0 + 2gu_0))}| \leqslant \sum_{i \in I^2_{N,\alpha}} |1 - e^{-\beta\Phi(x_i - (y_0+2gu_0))}| + |1 - e^{-\beta \sum_{i \in I^1_{N,\alpha}}^{N} \Phi(x_i - (y_0+2gu_0))}| \tag{32}$$

Using the Newton–Leibniz formula we obtain the following inequalities for the second term on the right side in (32)

$$\begin{aligned} &\left|1 - \exp\left\{-\beta \sum_{i \in I^1_{N,\alpha}} \Phi(x_i - (y_0+2gu_0))\right\}\right| \\ &\quad = \beta \int_0^1 \left|\sum_{i \in I^1_{N,\alpha}} \Phi(x_i - (y_0+2gu_0))\right| \\ &\quad\quad \times \exp\left\{-\beta s \sum_{i \in I^1_{N,\alpha}} \Phi(x_i - (y_0+2gu_0))\right\} ds \\ &\quad \leqslant \beta MN \int_0^1 e^{\beta s MN}\, ds \leqslant \beta MN e^{\beta MN} \end{aligned} \tag{33}$$

In the last two inequalities we used the obvious relation $N \geqslant |I^1_{N,\alpha}|$.

It follows from (32) and (33) that

$$\begin{aligned} &\left|1 - \exp\left\{-\beta \sum_{i=1}^{N} \Phi(x_i - (y_0+2gu_0))\right\}\right| \\ &\quad \leqslant \sum_{i \in I^2_{N,\alpha}} |1 - \exp\{-\beta\Phi(x_i - (y_0+2gu_0))\}| + \beta MN e^{\beta MN} \end{aligned}$$

Newt we obtain

$$J_N(\beta) \leqslant 2^N N(2g)^{(N-1)d} J_1(\beta) + \beta 2^N MN(2g)^{Nd} e^{\beta MN} \tag{34}$$

combining (27), (29), (31), and (34) we obtain

$$\begin{aligned} V(P_{\tau_1}, P_{\tau_2}) \leqslant e^{\beta MK_0 a + \beta M + \mu} \exp\{3(2g)^d e^{\beta MK_0 a + \beta M + \mu}\} \\ \times [J_1(\beta) + \beta M(2g)^d] \end{aligned} \tag{35}$$

Let

$$F_\beta(x) = |1 - e^{-\beta\Phi(x - (y_0+2gu_0))}|$$

for $x \in \bar{G}_0$ then

$$\lim_{\beta \to 0} F_\beta(x) \leqslant \begin{cases} 0, & \text{if} \quad x \neq y_0 + 2gu_0 \\ 1, & \text{if} \quad x = y_0 + 2gu_0 \end{cases}$$

Note that $\lim_{\beta \to 0} F_\beta(y_0 + 2gu_0) = 0$ in the case $\Phi(0) < \infty$. Since

$$F_\beta(x) \leqslant 2$$

for β small enough by the Lebesgue convergence theorem we obtain that

$$\lim_{\beta \to 0} J_1(\beta) = \lim_{\beta \to 0} \int_{\bar{G}_0} F_\beta(x)\, dx = 0$$

at fixed $\mu \in \mathbb{R}$. Moreover, it is easy to see that

$$\lim_{\beta \to 0} \sup_{y_0 \in \bar{G}_0} J_1(\beta) = 0 \tag{36}$$

Therefore for any τ_1, τ_2 satisfying conditions of the lemma

$$\lim_{\beta \to 0} V(P_{\tau_1}, P_{\tau_2}) = 0$$

Hence there exists $\beta_0(K_0) > 0$ such that for $\beta \leqslant \beta_0(K_0)$

$$V(P_{\tau_1}, P_{\tau_2}) < \frac{1}{2K_0 a} \tag{37}$$

Observe that $\beta_0(K_0)$ is independent on the boundary conditions τ_1 and τ_2 because of (36). Let $0 < \gamma < 1/2K_0 a$. Suppose that $\beta_0(K_0)$ is chosen such that

$$V(P_{\tau_1}, P_{\tau_2}) < \gamma$$

for every pair τ_1, τ_2 different from each other by one particle.

Next let τ_1 and τ_2 be such that $\tau_1(u_0) \neq \tau_2(u_0)$ and $\tau_1(u) = \tau_2(u)$ if $u \neq u_0$. It is easy to find a sequence of configurations $\sigma_k(u)$, $k = 0,...,$ $T = |\tau_1(u_0)| + |\tau_2(u_0)| - |\tau_1(u_0) \cap \tau_2(u_0)|$ such that

1. $\sigma_0(u_0) = \tau_1(u_0)$ and $\sigma_T(u_0) = \tau_2(u_0)$;
2. $|\sigma_k(u_0)| \leqslant \max\{|\tau_1(u_0)|, |\tau_2(u_0)|\}$ for every k;
3. for every k the configurations $\sigma_k(u_0)$ and $\sigma_{k+1}(u_0)$ differ by one particle;
4. for $u \neq u_0$ and every $k \sigma_k(u) = \tau_i(u)$.

Then for $\beta \leqslant \beta_0(K_0)$

$$V(P_{\tau_1}, P_{\tau_2}) \leqslant \sum_{i=1}^{T} V(P_{\sigma_i}, P_{\sigma_{i+1}}) \leqslant T\gamma < \frac{1}{a}$$

Finally, we consider a general case of τ_1 and τ_2. Let $\sigma_0,..., \sigma_n$, $n \leqslant a$, be a sequence of configurations with the properties

(1) $\sigma_0 = \tau_1$, $\sigma_n = \tau_2$;

(2) σ_k and σ_{k+1} differ only at one site, i.e. for every k there exists $u_k \in \partial 0$ such that $\sigma_k(u_k) \neq \sigma_{k+1}(u_k)$ and $\sigma_k(u) = \sigma_{k+1}(u)$ for all $u \neq u_k$;

(3) for every $u \in \partial 0$ and every $k \sigma_k(u) = \sigma_{k+1}(u)$ if $\tau_1(u) = \tau_2(u)$.

Let γ be as in (36). We set $\alpha' = T\gamma < 1/a$ and $k_u = \alpha'$. Then for $\beta \leqslant \beta_0(K_0)$

$$V(P_{\tau_1}, P_{\tau_2}) \leqslant \sum_{k=1}^{n} V(P_{\sigma_k}, P_{\sigma_{k+1}}) \leqslant \sum_{u \in \partial 0} k_u \delta(\tau_1(t+u), \tau_2(t+u))$$

$$\text{and} \qquad \alpha = \sum_{u \in \partial 0} k_u = \alpha' a < 1. \quad \blacksquare$$

7. UNIQUENESS

7.1. Uniqueness

The proof of Theorem 3 follows now from Lemmas 5 and 6 and the construction of the Section 4. As we can see from Lemma 5 and its proof the constants C and c are independent on β (see (22), (23)). Therefore $\rho = a^{b+1}c$ is independent on β as well and is equal to $1/1+\varepsilon$ where $\varepsilon > 0$ (see Section 5).

As in Lemma 6 we consider an arbitrary $K \geqslant 1$ and set $K_0 = (1/\chi) \ln K$. It can be seen from the proof of Lemma 6 that for every K_0 we can find $\beta_0 = \beta_0(K_0)$ such that (24) holds with $\sum_{u \in \partial 0} k_u < 1$. We chose K_0 such that $e^{\chi K_0} \geqslant \hat{K}$.

Then the condition of Theorem 4 are satisfied. $\blacksquare$

7.2. Existence and Uniqueness

Further in this section we assume that the stability condition (8) is satisfied. This ensures the finiteness of the denominator in (7) for any bounded V. Nevertheless the compactness conditions (Lemma 5) are not sufficient to provide the existence of at least one Gibbs field corresponding to the specification $\{P_{t,\tau}\}$. To have the existence we have to check a continuity property of the specification (see refs. 3 and 7).

Lemma 7. Let τ be a boundary configuration on $\mathbb{Z}^d\setminus\{t\}$ and let (τ_n) be a sequence of configurations on $\mathbb{Z}^d\setminus\{t\}$ which converges to τ. Then the measures P_{t,τ_n} weakly converge to the measure $P_{t,\tau}$.

We assume that the convergence $\tau_n \infty \tau$ means that $\tau_n(u) \to \tau(u)$ for every u.

Proof. We consider $t=0$ only. First we prove the convergence of corresponding partition functions. Namely

$$|Z_{\tau_n} - Z_\tau| \to 0 \qquad \text{as} \quad n \to \infty \tag{38}$$

We show that for every N

$$\int_{S^N} |e^{-\beta H(\sigma \mid \tau_n)} - e^{-\beta H(\sigma \mid \tau)}| \, \nu(d\sigma) \to 0 \tag{39}$$

as $n \to \infty$. Since $\tau_n \to \tau$ in the sense of the metric (15) then $|\tau_{n\partial t}| = |\tau_{\partial t}|$, where $|\tau_{n\partial t}|$ and $|\tau_{\partial t}|$ are the numbers of particles in the restriction to ∂t of the configurations τ_n and τ. Let T be this number, and let $\mathbf{y}'_n = (y_{n1},..., y_{nT})$ and $\mathbf{y}' = (y_1,..., y_T)$ be the sequences of co-ordinates in ∂t where particles of $\tau_{n\partial t}$ and $\tau_{\partial t}$ are localized. Recall that every coordinate is repeated in the sequence as many times as in the number of particles which are situated in this point. We can choose a numeration in the sequences $\mathbf{y}'_n$ and $\mathbf{y}'$ such that for every $i y_{ni} \to y_i$. Then (39) is equivalent to

$$\int_{\bar{G}_0^N} |e^{\beta \sum_{i=1}^T \sum_{j=1}^N \Phi(x_j - (y_{ni} + 2gu_i))} - e^{-\beta \sum_{i=1}^T \sum_{j=1}^N \Phi(x_j - (y_i + 2gu_i))}| \times dx_1 \cdots dx_N \to 0 \tag{40}$$

where $u_i \in \partial 0$ is such that $y_{ni}, y_i \in \tau(u_i)$. It is easy to see that

$$e^{-\beta \sum_{i=1}^T \sum_{j=1}^N \Phi(x_j - (y_{ni} + 2gu_i))} \to e^{-\beta \sum_{i=1}^T \sum_{j=1}^N \Phi(x_j - (y_i + 2gu_i))}$$

as $n \to \infty$. That gives (40) and (39).

To obtain (38) observe that for any $\varepsilon > 0$ there exists N_ε such that for all n

$$Z_{\tau_n} \leqslant \sum_{N=0}^{N_\varepsilon} e^{\mu N} \int_{S^N} e^{-\beta H(\sigma \mid \tau_n)} \nu(d\sigma) + \varepsilon \tag{41}$$

and

$$Z_\tau \leqslant \sum_{N=0}^{N_\varepsilon} e^{\mu N} \int_{S^N} e^{-\beta H(\sigma \mid \tau)} \nu(d\sigma) + \varepsilon \tag{42}$$

Now (38) follows from (40), (41) and (42) and the Lemma assertion is a simple consequence of (38). ▌

Next we give a new form of the main theorem, which is a little stronger. Recall that we require the stability condition.

Theorem 8. Let the specification be defined by the potential function Φ satisfying the conditions (i)–(vii) and the stability condition (8). Then for every $\mu \in \mathbb{R}$ there exists $\beta_1 > 0$ such that for $\beta \leqslant \beta_1$ there exists exactly one Gibbs measure m corresponding to the specification and such that for any volume V with $\operatorname{diam}(V) < \delta_0$

$$\mathsf{E} e^{\chi \, |\sigma_V|} < \infty$$

for all $\chi > 0$.

Proof. Let $0 < \chi_1 < \chi_2$. Then it follows from Theorem 3 and Lemma 7 that $|\mathfrak{A}_{\chi_1}| = 1$ if $\beta \leqslant \beta(\chi_1)$ and $|\mathfrak{A}_{\chi_2}| = 1$ if $\beta \leqslant \beta(\chi_2)$. Let $\mathfrak{A}_{\chi_i} = \{m_i\}$, $i = 1, 2$. For $\beta \leqslant \beta_0' = \min\{\beta(\chi_1), \beta(\chi_2)\}$ the inequalities

$$\mathsf{E}_{m_1} e^{\chi_1 \, |\sigma_V|} < \infty \qquad \text{and} \qquad \mathsf{E}_{m_2} e^{\chi_2 \, |\sigma_V|} < \infty$$

hold if $\operatorname{diam}(V) \leqslant \delta_0$. Here E_{m_i} means the expectation with respect to the measure m_i. Since $\chi_1 < \chi_2$ we have

$$\mathsf{E}_{m_2} e^{\chi_1 \, |\sigma_V|} \leqslant \mathsf{E}_{m_2} e^{\chi_2 \, |\sigma_V|}$$

The equality $|\mathfrak{A}_{\chi_1}| = 1$ implies that $m_1 = m_2$. It means that there exists the unique measure m for $\beta \leqslant \max\{\beta(\chi_1), \beta(\chi_2)\}$. Therefore there exists the unique measure in the region

$$\beta \leqslant \beta_1 = \sup_{\chi \geqslant 0} \beta(\chi) \quad ▌$$

2.3. Uniqueness Region

The proposition below gives a lower estimate of the function $\beta_1 = \beta_1(\mu)$ for negative values of μ.

Proposition 9. For any

$$\alpha < -\frac{1}{M(aK_0+1)}$$

the inequality

$$\lim_{\mu \to -\infty} \frac{\beta_1(\mu)}{\alpha\mu} \geqslant 1$$

holds.

Proof. The proof follows from (35), since $V(P_{t,\tau_1}, P_{t,\tau_2})$ has to be less than $1/2a$ (see (37)). ▌

For non-negative potential functions $\Phi(x) \geqslant 0$ the uniqueness region has the part where the uniqueness property holds for all temperatures.

Lemma 10. If $\Phi(x) \geqslant 0$ for any $x \in \mathbb{R}^d$ then for all $\beta \geqslant 0$ and for any $\tau_1, \tau_2 \in S^{\mathbb{Z}^d \setminus \{t\}}$ the following inequality is valid

$$V(P_{t,\tau_1}, P_{t,\tau_2}) < \text{const}(\exp\{e^{\mu}(\delta_0)^d\} - 1)$$

Proof. In the formula for the variation distance

$$V(P_{t,\tau_1}, P_{t,\tau_2}) = \sum_{N=0}^{\infty} \int_{S^N} |p_{t,\tau_1}(\sigma) - p_{t,\tau_2}(\sigma)| \, \nu(d\sigma)$$

we can replace the difference by the sum:

$$\begin{aligned} V(P_{t,\tau_1}, P_{t,\tau_2}) &\leqslant \sum_{i=1}^{2} \sum_{N=0}^{\infty} \int_{S^N} p_{t,\tau_i}(\sigma) \, \nu(d\sigma) \\ &\leqslant \sum_{i=1}^{2} \sum_{N=1}^{\infty} e^{\mu N} \int_{S^N} e^{-\beta H(\sigma \mid \tau_i)} \nu(d\sigma) \end{aligned}$$

If $\Phi(x) \geqslant 0$ then $H(\sigma(t) \mid \tau) \geqslant 0$ and we then obtain

$$\begin{aligned} V(P_{t,\tau_1}, P_{t,\tau_2}) &\leqslant \text{const} \sum_{i=1}^{2} \sum_{N=1}^{\infty} e^{\mu N} \int_{S^N} \nu(d\sigma) \\ &= \text{const}(\exp\{e^{\mu}(\delta_0)^d\} - 1) \end{aligned}$$ ▌

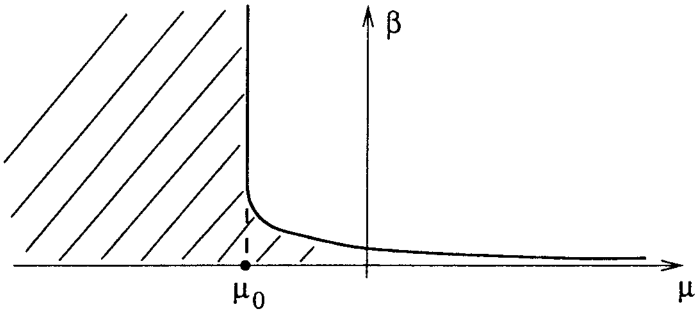

Fig. 2. The uniqueness region for the case of non-negative potential Φ.

It follows from this lemma that we can choose μ_0 such that for all $\mu \leqslant \mu_0$ and for all $\beta \geqslant 0$ the variance distance $V(P_{t, \tau_1}, P_{t, \tau_2})$ will be sufficiently small. Thus, in the case of repulsive interactions we have the uniqueness of Gibbs fields for all temperatures β provides $\mu \leqslant \mu_0$ (see Fig. 2).

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